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Quantum entanglement of particles on a ring with fractional statistics

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In this paper we investigate the von Neumann entropy in the ground state of one-dimensional anyonic systems with the repulsive interaction. Based on the Bethe-ansatz method, the entanglement properties for the arbitrary statistical parameter ($0 \leq \kappa \leq 1$) are obtained from the one-particle reduced density matrix in the full interacting regime. It is shown that the entanglement entropy increases with the increase in the interaction strength and statistical parameter. The statistic parameter affects the entanglement properties from two aspects: renormalizing of the effective interaction strength and introducing an additional anyonic phase. We also evaluate the entanglement entropy of hard-core anyons for different statistical parameters in order to clarify solely the effect induced by the anyonic phase.

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I. INTRODUCTION

In recent years quantum entanglement has attracted more and more attentions because it is not only a key resource in quantum information theory but also an essential concept in condensed matter physics [1]. It is well known that a great deal of properties of quantum many-body systems are closely related with the entanglement between particles, and the study of which is very important for both the system composed of distinguishable particles and the system composed of identical particles. However, unlike the system of distinguishable particles, for which there are various quantities to define and measure entanglement, the definition and quantification of entanglement between identical particles is still not very clear. Fortunately, the von Neumann entropy introduced by reduced density matrix is a good quantity to describe quantum entanglement for systems consisting of identical particles [2, 3, 4, 5, 6, 7, 8]. The entanglement entropy of two identical bosons or fermions in abstract wave functions has been studied by the Schmidt decomposition [2, 3, 4, 5], while the entanglement between two identical interacting trapped atoms in a continuous system was studied in Refs.[6, 7, 8]. It is shown that the statistical properties of identical particles play very important roles in their entanglement behaviors.

As a natural generalization of boson and fermion, anyon was proposed to describe the particle obeying fractional statistics and has been a subject of great interest in the past decades [9, 10, 11, 12, 13, 14, 15]. Although the concept of anyon arises originally from two-dimensional systems [12, 13, 14] related to fractional quantum Hall effect (FQH) and high-temperature superconductivity [9, 10, 11], the study of 1D anyonic model has attracted great theoretical interest [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. Motivated by possible experiments with cold atoms to simulate the creation and manipulation of anyons [31, 32, 33], and the possibility of performing topological quantum computation [34], the properties of 1D anyons are under current research focus. Several studies have been devoted to 1D anyons with a δ-function potential interaction [16, 17, 18, 19, 20, 21] and the limiting cases of hard-core anyons [23, 24, 25, 26, 27, 28, 29]. Particularly, the 1D interacting anyon model is exactly solvable by the Bethe-ansatz method as firstly found by Kundu [16]. Despite the intensive studies, the entanglement properties in anyonic systems are rarely studied except for the model in the hard-core limit [24].

In this work, we investigate the entanglement properties of a continuous system composed of N anyonic particles with repulsive contact interaction on a ring of length L by calculating the von Neumann entropy of the single-particle reduced density matrix. In general, it is hard to calculate the von Neumann entropy of a continuous many-body system analytically. So far, most of the studies focus on the two-particle system [6, 7, 8]. The integrability of the exactly solvable many-body system provides us the possibility to study the entanglement properties of a many-body system analytically. Based on the Bethe ansatz solution of the interacting anyonic model [16, 17], we evaluate the one-particle reduced density matrix firstly and then obtain the von Neumann entropy for the arbitrary statistical parameter in the full interacting regime.

This paper is organized as follows. In section II, we first give a brief introduction to the model and formulate the method. In section III, we first consider the Bose limit and focus on the effect of interacting strength on the von Neumann entropy. In section IV, we deal with the general anyonic case and discuss the effect of statistical parameter κ on the entanglement properties by calculating the von Neumann entropy for different κ. A brief summary is given in Section V.

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II. MODELS AND METHODS

The second quantized Hamiltonian for the one-dimensional anyonic system is formulated as

$$\begin{aligned}\mathcal{H}_A &= -\frac{\hbar^2}{2m}\int_0^L dx\Psi_A^\dagger \frac{\partial^2}{\partial x^2}\Psi_A \\ &+\frac{g_{1D}}{2}\int_0^L dx\Psi_A^\dagger(x)\,\Psi_A^\dagger(x)\,\Psi_A(x)\,\Psi_A(x)\,,\end{aligned} \quad (1)$$

in which m is the mass of anyons and g_{1D} denotes interacting strength between anyons [16, 17]. Here the field operators obey anyonic commutation relations $\Psi_A^\dagger(x_1)\Psi_A^\dagger(x_2) = e^{i\kappa\pi\epsilon(x_1-x_2)}\Psi_A^\dagger(x_2)\Psi_A^\dagger(x_1)$, and $\Psi_A(x_1)\Psi_A^\dagger(x_2) = \delta(x_1 - x_2) + e^{-i\kappa\pi\epsilon(x_1-x_2)}\Psi_A^\dagger(x_2)\Psi_A(x_1)$ with $\epsilon(x-y) = 1,-1,0$ for $x > y, x < y$, and $x = y$ respectively. The model (1) is known to be exactly solvable [16]. The parameter κ characterizes the statistical property of the anyonic system with $\kappa = 0$ and $\kappa = 1.0$ corresponding to Bose statistics and Fermi statistics respectively. The dependence of entanglement properties on both the interaction constant g_{1D} and statistical parameter κ $(0 \leq \kappa \leq 1)$ will be considered.

The eigenvalue problem of Hamiltonian (1) can be reduced to the quantum mechanical problem of N anyons with δ interaction [16, 17]

$$H\psi(x_1, ..., x_N) = E\psi(x_1, ..., x_N) \quad (2)$$

with

$$H = -\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + 2c\sum_{1\leq i\leq j\leq N}^{N}\delta(x_i - x_j), \quad (3)$$

where the natural unit is used and $c = mg_{1D}/\hbar^2$ $(c > 0)$ is a dimensionless interaction constant.

In terms of the exact ground-state wavefunction ψ, the single-particle reduced density matrix is defined as

$$\hat{\rho}_1 = \mathrm{Tr}_{2,3,...,N}|\psi\rangle\langle\psi|, \quad (4)$$

where the trace means to do integrations over all the position coordinates except one of them. The single-particle entanglement is quantified by the von Neumann entropy as

$$S = -\mathrm{Tr}(\hat{\rho}_1 \log_2 \hat{\rho}_1), \quad (5)$$

where $\hat{\rho}$ is the one-particle reduced density matrix with the normalization condition $\mathrm{Tr}\hat{\rho} = 1$. In coordinate representation, the one-particle reduced density matrix is expressed as

$$\begin{aligned}&\rho_1(x, x') \\ &= \frac{\int_0^L dx_2...dx_N[\psi^*(x, x_2, ...x_N)\psi(x', x_2, ..., x_N)]}{\int_0^L dx_1...dx_N|\psi(x_1, x_2, ..., x_N)|^2}.\end{aligned} \quad (6)$$

Obviously the (6) is Hermitian, i.e., we have $\rho_1^*(x, x') = \rho_1(x', x)$. The eigen- equation of the one-particle reduced density matrix (6) is

$$\int_0^L dx'\rho_1(x, x')\phi_\eta(x') = \lambda_\eta\phi_\eta(x), \quad (7)$$

where λ_η are the occupation numbers for natural orbitals $\phi_\eta(x)$ which form a complete and orthonormal set of functions. The one-particle reduced density matrix is diagonal in the basis of natural orbitals and $\sum_{\eta=1}^{\infty}\lambda_\eta = 1$. In terms of eigenvalues λ_η and the eigenfunctions $\phi_\eta(x)$, we can rewrite

$$\rho_1(x, x') = \sum_{\eta=1}^{\infty}\lambda_\eta\phi_\eta(x)\phi_\eta^*(x')$$

and the von Neumann entropy then reads

$$S = -\sum_{\eta=1}^{\infty}\lambda_\eta \log_2 \lambda_\eta. \quad (8)$$

From the above scheme, we can understand the difficulties which prevent us from studying the entanglement properties of a many-body system. First, the calculation of the ground-state wavefunction for a many-body system is generally difficult except for some exactly solvable systems. Furthermore, even though the exact many-body wave function is constructed, the calculation of the reduced density matrix for a large system remains a difficult task due to the time consuming to calculate multi-dimensional integrals.

III. THE DEPENDENCE OF ENTANGLEMENT ON THE INTERACTION STRENGTH

In this section, we shall focus on the Bose limit and study the dependence of entanglement on the interaction strength. The effect of fractional statistics will be discussed in the next section. In the Bose limit ($\kappa = 0$), the model is reduced to the well-known Lieb-Linger model [35]. In this case the field operators $\Psi_A^\dagger(x)$ and $\Psi_A(x)$ obey boson commutation relations and the wavefunction $\psi(x_1, ..., x_N)$ satisfies exchange symmetry. In the scheme of Bethe-ansatz method [35], the many-particle wave function can be formulated as

$$\begin{aligned}\psi(x_1, ..., x_N) &= \sum_Q \theta(x_{q_N} - x_{q_{N-1}})...\theta(x_{q_2} - x_{q_1}) \\ &\times\varphi_Q(x_{q_1}, x_{q_2}, ..., x_{q_N}),\end{aligned} \quad (9)$$

where Q labels the region $0 \leq x_{q_1} \leq x_{q_2} ... \leq x_{q_N} \leq L$, in which $q_1, q_2, ..., q_N$ is one of the permutations of $1, 2, ..., N$, Σ_Q sums over all permutations and $\theta(x - y)$ is the step function. Here $\varphi_Q(x_{q_1}, x_{q_2}, ..., x_{q_N})$ takes the Bethe-ansatz type

$$\varphi_Q(x_{q_1}, ..., x_{q_N}) = \sum_P[A_{p_1p_2...p_N}\exp(i\Sigma_j(k_{p_j}x_{q_j}))] \quad (10)$$

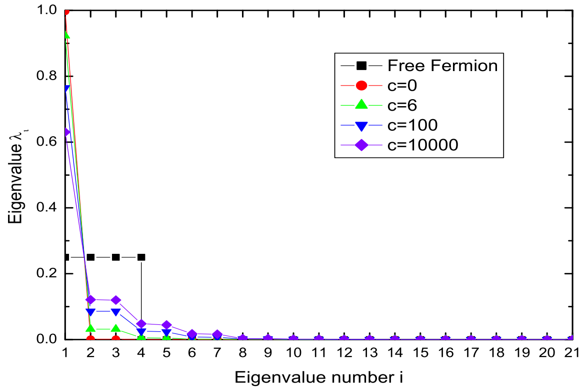

FIG. 1: The occupation numbers of the interacting Bose system for different interaction strength c.

with $A_{p_1 p_2 \dots p_N} = \varepsilon_P \prod_{j<l}^{N} (ik_{p_l} - ik_{p_j} + c)$ and k_{p_j} is a set of quasi-momentums determined by the Bethe-ansatz equations. Here $p_1, p_2, ..., p_N$ means one of permutations of $1, 2, ..., N$, and ε_P denotes a $+$ $(-)$ sign associated with even (odd) permutation of P. Using the periodical boundary condition, we can obtain Bethe-ansatz equations [35] whose logarithmic forms are formulated as

$$k_j L = 2n_j \pi - \sum_{l=1(l\neq j)}^{N} 2 \arctan\Big(\frac{k_j - k_l}{c}\Big), \tag{11}$$

where n_j is a set of integers to determine the eigenstates and for the ground state $n_j = (N+1)/2 - j$ $(1 \leq j \leq N)$. The energy of the system is $E = \sum_{j=1}^{N} k_j^2$ and the total momentum is $k = \sum_{j=1}^{N} k_j$.

By numerically solving the Bethe-ansatz equations (11), we can obtain the exact ground-state wavefunction, and then the one-particle reduced density matrix (6). Solving the eigenvalue problem (7) numerically, we can get a series of λ_i and then determine the entanglement entropy. For simplicity, we shall discuss the many-body system with $N = 4$ in the following context. In Fig. 1, we show the occupation numbers for the interacting Bose system composed of four identical bosons versus different interaction strength c. At $c = 0$, only the lowest nature orbital is occupied which means all the bosons condensate to the ground state. The occupation number of the lowest natural orbital λ_1 decreases with the increase of the interaction strength, accompanying with the increase of occupation numbers of higher natural orbitals. Our numerical results for the dependence of ground-state entanglement S on the interaction strength are shown in Fig. 2. In order to exhibit the change with c in a wide range, the logarithm coordinate for c is used in this figure. It is shown that the entanglement entropy changes monotonically with the change of interaction constant. When there is no interaction between bosons ($c = 0$), no entanglement exists in the system.

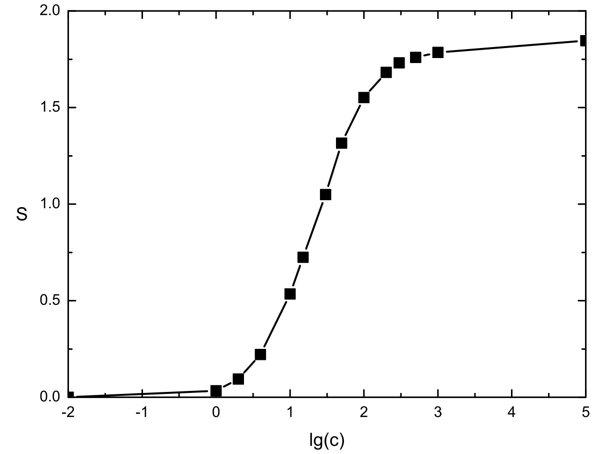

FIG. 2: The entanglement entropy S versus the logarithm of the repulsive interaction constant c for a system with 4 particles. One can see that S monotonically and smoothly increases from zero to about 1.846 with the growth of c from zero to the infinity limit.

Along with the growth of interaction constant, the entanglement entropy S increases slowly in the weakly interacting regime, and then goes sharply to 1.74. When the interaction gets close to the strongly interacting regime, the entanglement entropy S slowly approaches to about 1.846, which is smaller than 2. Our result is accordant with [4, 7], in which we note the entanglement entropy of N identical boson-particle system ranges from $S = 0$ for free-boson state to a maximum S in the infinitely repulsive limit which is smaller than $S = \log_2 N$. This can be understood as follows: when there is no interaction between particles, no correlation exists between bosons and the occupation number of lowest-energy state is N; while in the strong interaction limit ($c \to \infty$), particles will be prevented from occupying the same state and higher-energy states should be occupied.

IV. THE DEPENDENCE OF ENTANGLEMENT ON THE ANYONIC PARAMETER

Now we turn to the dependence of ground state entanglement entropy on the statistics. For anyonic system the many-body wave function shall satisfy the generalized symmetry

$$\psi(..., x_i, ..., x_j, ...) = e^{-i\theta} \psi(..., x_j, ..., x_i, ...), \tag{12}$$

where the anyonic phase

$$\theta = \kappa\pi \left[\sum_{k=i+1}^{j} \epsilon(x_i - x_k) - \sum_{k=i+1}^{j-1} \epsilon(x_j - x_k) \right].$$

for $i < j$. Considering the symmetry under coordinates reflection, we confine κ to $[0, 1]$ in the present paper. The

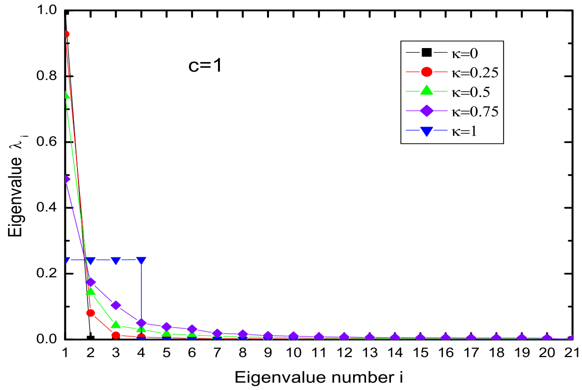

FIG. 3: The occupation numbers for different anyonic parameter κ with $c = 1$.

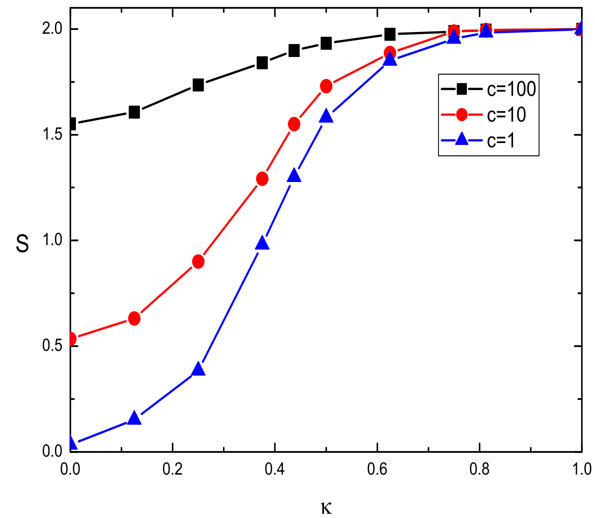

FIG. 4: (Color Online) The entanglement entropy versus the statistics parameter κ for the anyonic system with $c = 1$, $c = 10$ and $c = 100$, respectively.

wavefunction of anyons takes a similar form with that of bosons ($\kappa = 0$) [16, 17]

$$\psi_A(x_1, ..., x_N) = \sum_Q \theta(x_{q_N} - x_{q_{N-1}})...\theta(x_{q_2} - x_{q_1}) \times \phi_A \varphi_Q(x_{q_1}, x_{q_2}, ..., x_{q_N}), \quad (13)$$

where ϕ_A is an additional anyonic phase part

$$\phi_A = \exp(-i\frac{\kappa\pi}{2} \sum_{q_i<q_j} \epsilon(x_{q_i} - x_{q_j})) \quad (14)$$

and $\varphi_Q(x_{q_1}, x_{q_2}, ..., x_{q_N})$ has the same form as that of Lieb-Liniger Bose gas (10) except that now we have $A_{p_1 p_2 \ldots p_N} = \varepsilon_P \prod_{j<l}^N (ik_{p_l} - ik_{p_j} + c')$ with

$$c' = c/\cos(\kappa/2). \quad (15)$$

Similarly, the quasi-momenta k_i is determined by the Bethe ansatz equations

$$k_j L = 2n_j\pi - \sum_{l=1(l\neq j)}^{N} 2\arctan\Big(\frac{k_j - k_l}{c'}\Big), \quad (16)$$

under the twisted boundary condition $\psi_A(0, x_2, ..., x_N) = e^{i\kappa\pi(N-1)}\psi_A(L, x_2, ..., x_N)$. The Bethe-ansatz equations have the same form as that for Lieb-Liniger Boson gas (11) if we replace c with the renormalized interaction constant c'. According to (15) the effective interaction between anyons depends on the statistical parameter κ, which increases with the increase of κ and approaches ∞ in the fermionic limit ($\kappa \to 1.0$).

With the same procedure as presented in the above section we obtain the one-particle entanglement entropy for various c. As a concrete example, in Figure 3 we display the change of occupation numbers with different statistical parameters at a fixed interaction strength $c = 1$. The dependence of entanglement entropy on the statistical parameters κ is displayed in Figure 4 for $c = 1$, 10 and 100. It is shown that the entanglement entropy increases with the increasing of anyonic parameter κ and interaction strength c. For the case of $c = 10$, as $\kappa = 0$ the system reduces to Lieb-Liniger gas and the entanglement entropy S is about 0.535; while as $\kappa = 1/2$, S increases to about 1.730 and reaches 2 in the Fermi limit with $\kappa = 1.0$. The similar behaviors are displayed for $c = 1$ and $c = 100$. For different c the entanglement entropy S converges to $\log_2 N$ when $\kappa = \pi$ ($S_{max} = 2.0$ here).

In order to understand why and how κ affects the entanglement entropy, we compare the wavefunction of anyonic gas with that of Bose gas. According to (15) and (11), it is easy to find that the set of quasi-momenta k_j depend on c' and the wave function φ_Q changes along with the change of κ. In addition the phase factor ϕ_A includes the statistical parameter obviously. So the influence of statistical parameter κ on entanglement contains two parts: one coming from the phase ϕ_A and the other coming from wavefunction φ_Q through the renormalization of effective interaction strength c'.

Finally in order to clarify solely the dependence of entanglement entropy on the part of anyonic phase ϕ_A, we investigate the hard-core anyons, which can be studied using the anyon-fermion mapping method [23, 24]. In this situation the effective interaction $c' = \infty$ and the system has the same set of quasi-momenta k_j whatever the statistical parameter κ is. The fractional statistics only contributes a phase ϕ_A in the wavefunction. According to the numerical result shown in Figure 5, the entanglement entropy of hard-core anyonic gas increases monotonically when the statistical parameter κ changes

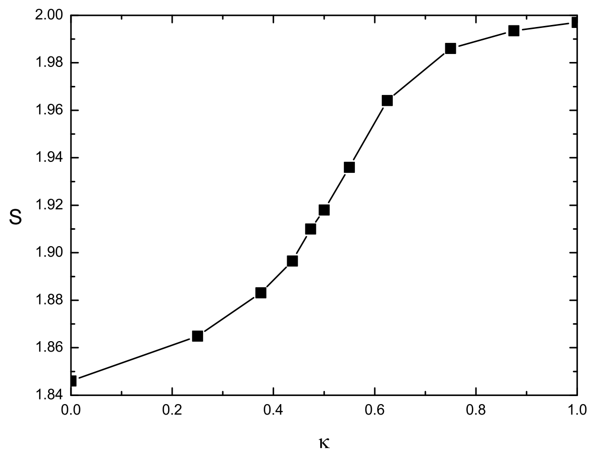

FIG. 5: The entanglement entropy versus the statistics parameter κ for the hard-core anyonic gas with $N = 4$.

from the Bose limit $\kappa = 0$ to the Fermi limit $\kappa = 1.0$.

V. SUMMARY

In summary, we have investigated the ground-state entanglement of the 1D anyonic system with repulsive interaction by calculating the one-particle von Neumann entropy in the full interacting regime ($0 \leq c \leq \infty$) for arbitrary statistical parameter ($0 \leq \kappa \leq 1.0$). Using the Bethe-ansatz method, we obtain the exact ground-state wavefunction, and thus the one-particle reduced density matrix and the von Neumann entropy. In the Bose limit ($\kappa = 0$) the entanglement entropy increases monotonically with the increase of the interaction strength and approaches a maximum in the hard-core limit. While for anyonic system it is shown that the entanglement entropy increases monotonically both with the increase in the interaction strength and statistical parameter. The anyonic gas gets to the maximum entanglement entropy $\log_2 N$ in the Fermi limit ($\kappa = 1.0$). The statistical parameter κ affects the entanglement properties of the anyonic system by renormalizing the effective interaction strength and introducing an additional anyonic phase in the wavefunction. The influence of anyonic phase on the entanglement is also clarified by evaluating the entanglement entropy of hard-core anyons for different statistical parameters.

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